(Defn) Power Series:- A series of the type

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1(z-z_1) + \cdots + a_n(z-z_0)^n + \cdots$$

is called the Power Series where variable z and the constants a_n, z_0 are complex numbers and a_n is independent of z.

(Defn) Circle and Radius of Convergence of Power Series:- The circle $|z| = R$ which includes in its interior $|z| < R$ all the values of z for which the power series $\sum a_n z^n$ converges, is called the Circle of Convergence of the series. The radius R of this circle is called the radius of convergence of the series.

Q No> Every convergent sequence is a Cauchy sequence.

Proof:- Let $\langle z_n \rangle$ converge to z_0. Then to each $\epsilon > 0$ there exists n_0 such that

$$|z_n - z_0| < \frac{\epsilon}{2} \text{ for all } n \geq n_0$$

Hence if $m \geq n_0$, $n \geq n_0$,

then $|z_m - z_0| < \frac{\epsilon}{2}$ and $|z_n - z_0| < \frac{\epsilon}{2}$ — ①

$$\therefore |z_m - z_n| = |z_m - z_0 + z_0 - z_n|$$

$$\leq |z_m - z_0| + |z_0 - z_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ by } ①$$

Hence, $\langle z_n \rangle$ is a Cauchy sequence.

Q No→ State and Prove Cauchy - Hadamard Theorem

Statement - For every Power Series $\sum_{n=0}^{\infty} a_n z^n$ there exist a number R, $0 \le R < \infty$, called the radius of convergence with the following Properties

(i) The Series Converges absolutely for every $|z| < R$

(ii) If $0 \le \rho < R$, the Series Converges uniformly for $|z| \le \rho$.

(iii) If $|z| > R$, the terms of the Series are unbounded and Series is consequently divergent

Proof:- We shall Show that the theorem holds if we choose R according to the formula,

$$\frac{1}{R} = \lim_{n\to\infty} \sup |a_n|^{1/n} \quad \text{——— } (1)$$

(i) Let R be as defined above and let $|z| < R$. Then there exists ρ Such that $|z| < \rho < R$ Which implies that, $\frac{1}{\rho} > \frac{1}{R}$

By the definition of limit Superior and (1), there exists a +ve integer n_0 Such that

$$|a_n|^{1/n} < \frac{1}{\rho}, \text{ i.e. } |a_n| < \frac{1}{\rho^n} \text{ for all } n \ge n_0$$

It follows that,

$$|a_n z^n| < \left(\frac{|z|}{\rho}\right)^n \text{ for large } n.$$

But the Series $\sum_{n=0}^{\infty} \left(\frac{|z|}{\rho}\right)^n$ is a geometric Series of Common ratio less than 1.

$\therefore$ Since $|z| < \rho$ and Consequently it Converges Then by Comparison test, the series $\sum |a_n z^n|$ also Converges and hence the Power Series $\sum a_n z^n$ Converges absolutely for all z with $|z| < R$.

(ii) Let $0 \leq \rho < R$, To Show that the Series Converges uniformly for $|z| \leq \rho$. We Choose ρ' Such that $\rho < \rho' < \infty$.

As in (i), we have

$$|a_n| \leq \frac{1}{\rho^n} \text{ for } n \geq n_0$$

$$\text{Hence } |a_n z^n| \leq \left(\frac{|z|}{\rho}\right)^n < \left(\frac{\rho'}{\rho}\right)^n \quad [\because |z| \leq \rho < \rho']$$

Now, the series $\sum \left(\frac{\rho'}{\rho}\right)$ of +ve Consttn. is Convergent being a geometric Series of Common ratio $\rho'/\rho < 1$. Hence by Weierstrass M-test the Power series $\sum a_n z^n$ Converges uniformly for $|z| \leq \rho < R$.

(iii) If $|z| > R$, We Choose ρ Such that $R < \rho < |z|$.

Then, $\frac{1}{R} > \frac{1}{\rho}$ Since, $\frac{1}{R} = \lim_{n \to \infty} \sup |a_n|^{1/n}$, it follows that there are arbitrary large n Such that,

$$|a_n|^{1/n} > \frac{1}{\rho}, \text{ i.e. } |a_n| > \frac{1}{\rho^n}$$

and Consequently $|a_n z^n| > (|z|/\rho)^n$ for infinitely many n. Hence, the terms of the Series are unbounded. Accordingly, the Series is divergent.

QNo. Let $\sum_{n=0}^{\infty} a_n z^n$ be a Power Series and let $\sum_{n=1}^{\infty} n a_n z^{n-1}$ be the Power Series obtained by differentiating the first series term by term. Then the derived series has the same radius of convergence as the original series.

Proof: Let R and R' be the radii of convergence of the series

$$\sum_{n=0}^{\infty} a_n z^n \text{ and } \sum_{n=1}^{\infty} n a_n z^{n-1} \text{ respectively.}$$

Then they are given by the formula,

$$\frac{1}{R} = \overline{Lt}\, |a_n|^{1/n} \text{ and } \frac{1}{R'} = \overline{Lt}\, n^{1/n} |a_n|^{1/n}.$$

Theorem will be proved if we show that $Lt\, n^{1/n} = 1$.

To prove this, let $n^{1/n} = 1 + h_n$ so that

$$n = (1+h_n)^n = 1 + nh_n + \frac{n(n-1)}{1\cdot 2} h_n^2 + \cdots + h_n^n.$$

Hence $n > \frac{n}{2}(n-1) h_n^2$ or, $h_n^2 < \frac{2}{n-1}$ so that $h_n \to 0$ as $n \to \infty$.

It follows that $\lim_{n\to\infty} n^{1/n} = 1$

$\therefore R = R'$.

Q.No→ Prove that a Power Series represents an analytic function inside its circle of Convergence.

Or, Q.No→ Prove that a Power Series in z represents an analytic function regular within its circle of Convergence.

Or, Q.No→ The Sum function $f(z)$ of the Power Series $\sum_{n=0}^{\infty} a_n z^n$ represents an analytic function inside its Circle of Convergence.

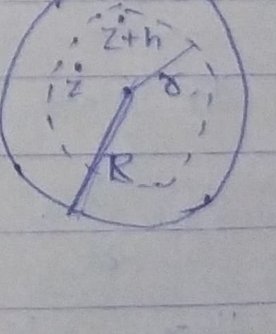
Proof:- Let R be the radius of Converges of any given Power Series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \text{ So that, we have}$$

$$\frac{1}{R} = \overline{Lt}\, |a_n|^{1/n}.$$

Let $\phi(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$. The radius of Converges of the series is also R. Suppose z is any Point within the circle of Convergence, So that $|z| < R$. Then there exist a +ve number r such that

$$|z| < r < R.$$



For Convenience, we write $|z| = \rho$, $|h| = \eta$. Then $\rho < r$. Also h may be so chosen that $\rho + \eta < r$.

Take z+h any Point such that

$$|z+h| \le r < R.$$

So, that $|a_n r^n| < M$ where M is finite & +ve

We then have

$$\left|\frac{f(z+h)-f(z)}{h} - \phi(z)\right| = \left|\sum_{n=0}^{\infty}\left[a_n\left\{\frac{(z+h)^n - z^n}{h} - nz^{n-1}\right\}\right]\right|$$

$$= \left|\sum_{n=0}^{\infty}\left[a_n\left\{\frac{n(n-1)}{1\cdot 2} z^{n-2}|h| + \cdots + h^{n-1}\right\}\right]\right|$$

$$\leq \sum_{n=0}^{\infty}|a_n|\left\{\frac{n(n-1)}{1\cdot 2}|z|^{n-2}|h| + \cdots + |h|^{n-1}\right\}$$

$$\leq \sum_{n=1}^{\infty}\frac{M}{r^n}\left\{\frac{n(n-1)}{1\cdot 2}\rho^{n-2}\eta + \cdots + \eta^{n-1}\right\}.$$

$$\leq \sum_{n=0}^{\infty}\frac{M}{r^n}\cdot\frac{1}{\eta}\left\{(\rho+\eta)^n - \rho^n - n\rho^{n-1}\eta\right\}.$$

$$= \frac{M}{\eta}\sum_{n=0}^{\infty}\left\{\left(\frac{\rho+\eta}{r}\right)^n - \left(\frac{\rho}{r}\right)^n - \frac{\eta n}{\rho}\left(\frac{\rho}{r}\right)^n\right\} \quad (1)$$

Now, $\displaystyle\sum_{n=0}^{\infty}\left(\frac{\rho+\eta}{r}\right)^n = 1 + \frac{\rho+\eta}{r} + \left(\frac{\rho+\eta}{r}\right)^2 + \cdots$

$$= \frac{1}{1-\frac{\rho+\eta}{r}} = \frac{r}{r-\rho-\eta}$$

and $\displaystyle\sum_{n=0}^{\infty}\left(\frac{\rho}{r}\right)^n = 1 + \frac{\rho}{r} + \left(\frac{\rho}{r}\right)^2 + \cdots = \frac{1}{1-\frac{\rho}{r}} = \frac{r}{r-\rho}.$

To sum, $\sum n\left(\frac{\rho}{r}\right)^n$, we put,

$$S = \frac{\rho}{r} + 2\left(\frac{\rho}{r}\right)^2 + 3\left(\frac{\rho}{r}\right)^3 + 4\left(\frac{\rho}{r}\right)^4 + \cdots$$

then, $\displaystyle S\frac{r}{\rho} = \left(\frac{\rho}{r}\right)^2 + 2\left(\frac{\rho}{r}\right)^3 + 3\left(\frac{\rho}{r}\right)^4 + \cdots$

∴ Subtracting, we get

$$S\left(1-\frac{\rho}{r}\right)=\frac{\rho}{r}+\left(\frac{\rho}{r}\right)^2+\left(\frac{\rho}{r}\right)^3+\cdots=\frac{\frac{\rho}{r}}{1-\frac{\rho}{r}}=\frac{\rho}{r-\rho}.$$

$$\therefore S=\frac{\rho r}{(r-\rho)^2}$$

Substituting these values in ①, we get

$$\left|\frac{f(z+h)-f(z)}{h}-\phi(z)\right|\leq\frac{M}{\eta}\left\{\frac{r}{r-\rho-\eta}-\frac{r}{r-\rho}-\frac{\eta r}{(r-\rho)^2}\right\}$$

$$=\frac{M}{\eta}\frac{r\cdot\eta^2}{(r-\rho-\eta)(r-\rho)^2}$$

$$=\frac{Mr\eta}{(r-\rho-\eta)(r-\rho)^2}$$

Which tends to zero as $\eta\to 0$

$$\therefore \lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=\phi(z)$$

~~It follows that f(z) has the derivative φ(z).~~

Again, Series the radius of Convergence of the derived Series.

$$\phi(z)=\sum_{n=1}^{\infty}na_nz^{-1}.$$

It follows that $f(z)$ has the derivative $\phi(z)$

$$\text{i.e. } f'(z)=\sum_{n=1}^{\infty}na_nz^{-1}.$$

So that $f'(z)$ is given on term by term differentiation of the series for $f(z)$

Thus $f(z)$ is analytic in $|z|<R$.

Ex. (1) Find the radii of convergence of the following series:-

(a) The Power series $\sum \frac{2^{-n}}{1+in^2} z^n$,

Solu^n:- Here, $a_n = \frac{2^{-n}}{1+in^2}$

$$\frac{1}{R} = \underset{n\to\infty}{Lt} |a^n|^{1/n} = \underset{n\to\infty}{Lt} \left|\frac{2^{-n}}{1+in^2}\right|^{1/n}$$

$$= \underset{n\to\infty}{Lt} \frac{2^{-1}}{|1+in^2|^{1/n}}$$

$$= \frac{1}{2} \underset{n\to\infty}{Lt} \frac{1}{(1+n^4)^{1/2n}} \quad \text{Since, } |1+in^2| = \sqrt{(1+n^4)}$$

$$= \frac{1}{4} \underset{n\to\infty}{Lt} \frac{1}{(n^{1/n})^2 \left\{1+\frac{1}{n^4}\right\}^{1/n}} = \frac{1}{2} \quad \therefore R = 2.$$

(b) The Power series $\sum \frac{2+in}{2^n} z^n$.

Solu^n Here, $a_n = \frac{2+in}{2^n}$

$$\frac{1}{R} = \underset{n\to\infty}{Lt} |a_n|^{1/n} = \underset{n\to\infty}{Lt} \left|\frac{2+in}{2^n}\right|^{1/n} = \underset{n\to\infty}{Lt} \frac{|2+in|^{1/n}}{|2^n|^{1/n}}$$

$$= \underset{n\to\infty}{Lt} \frac{(4+n^2)^{1/2n}}{2} = \underset{n\to\infty}{Lt} \frac{n^{1/n}\left(1+\frac{4}{n^2}\right)^{1/2n}}{2}$$

$$= \frac{1}{2} \quad \text{Since, } \underset{n\to\infty}{Lt} n^{1/n} = 1.$$

$\therefore$ R = 2 is the radius of convergence.